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MATERIAL AND SPATIAL REPRESENTATIONS OF THE CONSTITUTIVE RELATIONS OF DEFORMABLE MEDIA*

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The problem of giving a basis to material and spatial representations for the constitutive relations (CR) of media, their correspondence (equivalence) to each other, as well as the problem of the explicit resolution of implicit forms of CR (in material and spatial representations) are examined from the point of view of the general theory of constitutive relationships of the classical mechanics of a continuous medium, based on the principles of determinism and causality, locality, independence of the reference system, and the hypothesis of macrophysical determinacy.

Approaches based on the introduction of spatial-type tensors, used in an Euler description (/20-28/, say) are used in addition to the traditional approaches of the mechanics of a continuous medium that are in direct agreement with the macroscopic determinacy hypothesis /1, 2/, and expressed from the beginning, as a rule, in the terminology of the material-type tensors utilized in the Lagrange description of motion of a medium (see /2-7/, say), or explicitly understood by the connection with such tensors (for instance /8-19/). Numerous papers on plasticity that propose extrapolation of the CR, known for small deformations, by some method to the case of finite deformations in an Euler description are among them.

Important questions arise here, in principle: 1) is such extrapolation legitimate from the point of view of the general classical theory of CR, i.e., does the CR obtained agree, in principle, with the postulate of macroscopic determinability? (the example in /28/ is one of the modifications of such an erroneous inconsistency), 2) which is the spatial representation (Euler form) of the constructed or known material (Lagrange) CR and vice-versa? 3) if the CR is constructed in implicit form, especially in the spatial tensor

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terminology (for instance, in the form of a differential equation with objective derivatives), then is it uniquely solvable for the stress tensor (i.e., is the principle of determinism satisfied), and what is the procedure for such a solution (integration of the differential equations with objective derivatives)? Investigation of these questions could assist the analysis, ordering, and development of investigations on the construction of CR of complex media for finite strains in the direction of a sequential development of the theory of the experiment.

In this paper a set of general reduced forms of CR of the classical mechanics of a continuous medium is derived, including the CR of the Il'yushin postulate of macroscopic determinability, and the Noll CR and answers are obtained to the questions formulated above, illustrated by examples.

1. Tensor characteristics of mechanical processes. The equivalence of tensors and mappings. In the motion of a deformable medium with a Lagrange law of motion and the deformation affinor

$$\mathbf{x} = \mathbf{f}(\mathbf{a}, t), \quad \mathbf{A} = \nabla_a \mathbf{f} \equiv \mathbf{QX} \equiv \mathbf{YQ} \tag{1.1}$$

(where **a**, **x** are the radius-vector of a point of the medium in reference (undeformed) and actual (deformed) configurations, t is the time, ∇_a is the gradient operator on **a**, **Q** is orthogonal and **X**, **Y** are symmetric right and left polar expansion tensors of **A**), tensors of different ranks r above a basic (three-dimensional) vector space are used to describe the mechanical processes in a material neighbourhood of a point of the body: scalars $\varphi\ (r = 0)$, vectors $\mathbf{u}, \mathbf{z}\ (r = 1)$, tensors **U**, **Z** of the second $(r = 2)$ and higher ranks. Limiting ourselves here for simplicity to the cases $r \leqslant 2$ as in /6, 7, 16/, we separate the set of tensor characteristics $\varphi, \mathbf{u}, \mathbf{z}, \mathbf{U}, \mathbf{Z}$ that are transformed to the characteristics $\varphi_*, \mathbf{u}_*, \mathbf{z}_*, \mathbf{U}_*, \mathbf{Z}_*$ as the reference system changes by means of the formulas

$$\varphi_* \equiv \varphi, \quad \mathbf{u}_* \equiv \mathbf{u}, \quad \mathbf{z}_* \equiv \boldsymbol{\Theta}\mathbf{z}, \quad \mathbf{U}_* \equiv \mathbf{U}, \quad \mathbf{Z}_* \equiv \boldsymbol{\Theta}\mathbf{Z}\boldsymbol{\Theta}^T \tag{1.2}$$

where $\boldsymbol{\Theta} \equiv \boldsymbol{\Theta}(t)$ is the orthogonal tensor for the passage to the new reference system.

Examples of such quantities for φ, are the mass density, internal energy, and also scalar invariants of the remaining quantities in (1.2) or for **u** and **z** vectors of elementary material fibres in the reference configuration $\delta\mathbf{a}$ and actual configuration $\delta\mathbf{x}$, respectively, as well as eigenvectors of any tensors **U** and **Z** from (1.2), respectively, examples of **U** and **Z** are, respectively, right **X** and left **Y** distortion tensors from (1.1), the Cauchy $\mathbf{C} = \mathbf{X}^2$ and Finger $\mathbf{F} = \mathbf{Y}^2$ strain measures, the Green's $\mathbf{E}_\mathrm{I} = {}^1/_2(\mathbf{C} - \mathbf{I})$ and Almansi $\mathbf{E}_\mathrm{I} = {}^1/_2(\mathbf{I} - \mathbf{F}^{-1})$ strain tensors (**I** is the unit tensor), the material derivative $\mathbf{E}_\mathrm{I}^{\cdot}$ and the strain rate tensor **V**, the Piola-Kirchhoff $J\boldsymbol{\Sigma}_\mathrm{I} = J\mathbf{A}^{-1}\mathbf{S}\mathbf{A}^{-1T}$ and Cauchy $\mathbf{S}\ (J = |\det \mathbf{A}|)$ etc. /1-29/ stress tensors of the second kind.

The tensors **u**, **U**, etc, that are transformed into similar ones $(r \geqslant 1)$, are called materially (reference) oriented (or right, for brevity), and the tensors **z**, **Z** and similar ones $(r \geqslant 1)$ spatially oriented (or briefly, left). All the tensors $(r \geqslant 0)$ from (1.2) are called objective (in recent years the names "objective" or "independent of the reference system" or "indifferent" have often referred in the literature to just the left tensors (see /9-12, 16/, for instance). As the examples presented above show, they can be used for an objective description of mechanical processes: materially oriented from the aspect of the observer connected to a material particle (in the reference configuration), spatially oriented from the aspect of a spatial observer, and objective scalars from both aspects. Thus, the state of strain and stress in a material particle of a medium is described completely, from both aspects, by the right tensors **X** and $\boldsymbol{\Sigma}_\mathrm{I}$ (for the Lagrange description), and by the left tensors **Y** and **S** (for the Euler description), respectively.

Later, on the basis of the Lagrange (mateial, reference) description, we will consider all the objective tensors. Let $M^{(r)}$ and $S^{(r)}\ (r \geqslant 1)$ denote sets of right and left tensor processes of rank r while $M^{(0)} \equiv S^{(0)}$ is a set of objective scalars for a given material particle. It is clear that $M^{(r)}, S^{(r)}$ are only subsets of all the mechanical tensor characteristics (tensor processes).

Just like the well-known relations $\delta\mathbf{x} = \mathbf{A}\delta\mathbf{a}$, $\mathbf{Y} = \mathbf{QXQ}^T$, $\mathbf{S} = \mathbf{A}\boldsymbol{\Sigma}_\mathrm{I}\mathbf{A}^T$ the right and left tensors, describing the very same mechanical processes from the aspects mentioned, can be connected by equivalence relationships of the form

$$\mathbf{z} = \boldsymbol{\Delta}\mathbf{u}, \quad \mathbf{Z} = \boldsymbol{\Delta}\mathbf{U}\boldsymbol{\Lambda} \tag{1.3}$$

where $\boldsymbol{\Delta}$ and $\boldsymbol{\Lambda}$ are non-degenerate tensors of the second rank that are transformed in the same way as (1.2) as the reference system changes, according to the formulas

$$\boldsymbol{\Delta}_* \equiv \boldsymbol{\Theta}\boldsymbol{\Delta}, \quad \boldsymbol{\Lambda}_* \equiv \boldsymbol{\Lambda}\boldsymbol{\Theta}^T \tag{1.4}$$

The simplest case of relationships of the type (1.3), realized for $\boldsymbol{\Lambda} \equiv \boldsymbol{\Lambda}^T$ is examined in /29/. Without limiting ourselves to this case, we note that the tensors $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}$ satisfying (1.4) allow of a unique representation of the form

$$\Delta \equiv \Theta\Xi, \quad \Lambda \equiv \Upsilon Q^T \ (\Xi, \Upsilon \in M^{(2)}) \tag{1.5}$$

where Q is from (1.1). Therefore, selection of the tensors Δ, Λ determining the equivalence relationships (1.3) reduces to selecting the right tensors Ξ, Υ. If the tensors Δ, Λ (or Ξ, Υ) are determined by the strain affinor process $\mathbf{A}$ of this material particle

$$\Delta = l\,[\mathbf{A}], \ \Lambda = m\,[\mathbf{A}], \ \Xi = l_0\,[\mathbf{A}], \ \Upsilon = m_0\,[\mathbf{A}] \tag{1.6}$$

then their selection reduces to selecting the mappings l, m satisfying (1.4), or l_0, m_0, that take values in $M^{(2)}$.

Taking account of (1.4) or (1.5) for the selected Δ, Λ relationships (1.3) set up a one-to-one correspondence between the sets $M^{(r)}$ and $S^{(r)}$ of the tensor processes of rank r, and for any mapping $L_M : M^{(p)} \to M^{(q)}$ also induce the mapping action L_S: $S^{(i)} \to S^{(q)}$:

$$\begin{gathered} L_S[\varphi] = L_M[\varphi] \quad (p = q = 0), \quad L_S[\varphi] = \Delta L_M[\varphi] \quad (p = 0, \quad q = 1) \\ L_S[\varphi] = \Delta L_M[\varphi]\,\Lambda \quad (p = 0, \ q = 2), \quad L_S[\mathbf{z}] = L_M[\Delta^{-1}\mathbf{z}] \quad (p = 1, \ q = 0) \\ L_S[\mathbf{z}] = \Delta L_M[\Delta^{-1}\mathbf{z}] \quad (p = q = 1), \quad L_S[\mathbf{z}] = \Delta L_M[\Delta^{-1}\mathbf{z}]\,\Lambda \quad (p = 1, q = 2) \\ L_S[\mathbf{Z}] = L_M[\Delta^{-1}\mathbf{Z}\Lambda^{-1}] \quad (p = 2, \ q = 0) \\ L_S[\mathbf{Z}] = \Delta L_M[\Delta^{-1}\mathbf{Z}\Lambda^{-1}] \quad (p = 2, q = 1) \\ L_S[\mathbf{Z}] = \Delta L_M[\Delta^{-1}\mathbf{Z}\Lambda^{-1}]\,\Lambda \quad (p = q = 2) \end{gathered} \tag{1.7}$$

and conversely, by using formulas inverse to (1.7) for any $L_S : S^{(p)} \to S^{(q)}$ we obtain $L_M : M^{(p)} \to M^{(q)}$. In particular, if $L_M : M^{(r)} \to M^{(r)}$ is a material diferentiation operator, then by denoting it by a dot $(\cdot)^{\cdot}$ and its spatial analogue $L_S[\cdot]$ by $D[\cdot]$, we obtain

$$\begin{gathered} D\,[\varphi] \equiv \varphi^{\cdot} \ (r = 0), \ D\,[\mathbf{z}] \equiv \Delta\,(\Delta^{-1}\mathbf{z})^{\cdot} \equiv \mathbf{z}^{\cdot} - \mathbf{G}_\Delta \mathbf{z} \ \ (r = 1), \\ D\,[\mathbf{Z}] \equiv \Delta\,(\Delta^{-1}\mathbf{Z}\Lambda^{-1})^{\cdot}\,\Lambda \equiv \mathbf{Z}^{\cdot} - \mathbf{G}_\Delta \mathbf{Z} - \mathbf{Z}\mathbf{G}_\Lambda \quad (r = 2) \\ (\mathbf{G}_\Delta = \Delta^{\cdot}\Delta^{-1}, \quad \mathbf{G}_\Lambda = \Lambda^{-1}\Lambda^{\cdot}) \end{gathered} \tag{1.8}$$

By using relationships of the type (1.3) and (1.8) and taking (1.6) into account, on the basis of the left tensors $\mathbf{V}, \mathrm{S} \in S^{(2)}$ ($\mathbf{V} = \mathrm{sym}\,(\mathbf{A}^{\cdot}\mathbf{A}^{-1})$ is the strain rate tensor and S is the Cauchy stress tensor), tensor strain measures $\Psi \in M^{(2)}, \mathbf{E} \in S^{(2)}$ and stress measures $\Sigma \in M^{(2)}$ can be introduced by the equations

$$\Psi^{\cdot} = \Delta_1^{-1}\mathbf{V}\Lambda_1^{-1}, \quad D[\mathbf{E}] = \mathbf{V}, \quad \Sigma = \Delta_2{}^T\mathbf{S}\Lambda_2{}^T \tag{1.9}$$

where Δ_l, Λ_l $(l = 1, 2)$ are non-degenerate tensors satisfying relationships (1.4) and (1.6), $D_1[\cdot] \equiv D[\cdot]$ from (1.8) with $\Delta \equiv \Delta_1, \Lambda \equiv \Lambda_1$, and then $\mathbf{E} \equiv \Delta\Psi\Lambda$.

Formulas (1.8) and (1.9) include, as special cases, the known /2-27, 29/ derivatives of left vectors and tensors as well as the tensor strain and stress measures obtained by a different selection Δ_l, Λ_l $(l = 1, 2)$.

2. ***The non-dependence of the mappings and equations on the reference system.*** As a rule, mappings (functions, functionals and operators) independent of the reference system, i.e., the rules for giving (determining) them are identical in all reference systems, are used to describe the connections of the mechanical processes at a given material neighbourhood of a point of the body. Namely, the mapping $L : \Gamma \to \Pi$, where Γ and Π are sets of some sets $\gamma \in \Gamma$ and $\pi \in \Pi$ of scalar, vector, and tensor characteristics of mechanical processes, will be said to be independent of the reference system if, on chaning to an arbitrary new reference system (when $\gamma, \pi, \Gamma, \Pi, L$ are converted to $\gamma_*, \pi_*, \Gamma_*, \Pi_*, L_*$) we have $\Gamma_* = \Gamma, \Pi_* = \Pi$ (conservation of the domain of definition and the domain of values) as well as either of the following two (equivalent) assertions

$$L_* = L, \ \pi_* = L\,[\gamma_*], \ \forall\gamma \in \Gamma \tag{2.1}$$

Differentiation operators and integration operators with respect to time, and the operations of addition, multiplication of scalar, vector, and tensor quantities (processes) may be examples of such mappings for different Γ and Π. In cases when the arguments or values of the mapping L are converted according to definite formulas (for example, of the from (1.2) or (1.4)) on changing to a new reference system, then a universal canonical representation (general reduced form) is obtained successfully for the mapping L that is independent of the reference system. This example yields the following lemma.

Lemma 1. If relations (1.4) (or (1.5)) are satisfied and the mappings l and m (or l_0 and m_0) in (1.16) are independent of the reference system, then the mappings l_0 and m_0 (or l and m) are independent of the reference system and the following representations hold

$$\begin{gathered} l\,[\mathbf{A}] = \mathbf{Q}l\,[\mathbf{X}] \equiv \mathbf{Q}l_0\,[\mathbf{X}], \quad m\,[\mathbf{A}] = m\,[\mathbf{X}]\,\mathbf{Q}^T \equiv m_0\,[\mathbf{X}]\,\mathbf{Q}^T \\ l_0\,[\mathbf{A}] = l_0\,[\mathbf{X}] \equiv l\,[\mathbf{X}], \quad m_0\,[\mathbf{A}] = m_0\,[\mathbf{X}] \equiv m\,[\mathbf{X}] \end{gathered} \tag{2.2}$$

The operators L_M and L_S from (1.7) are of interest.

Lemma 2. The mapping $L = L_M : M^{(p)} \to M^{(q)}$ is independent of the reference system if and only if it is invariant shifts of the time argument

$$\pi(t) = L[\gamma(\tau)] \Rightarrow \pi(t+c) = L[\gamma(\tau+c)], \quad \forall c = \text{const}$$

For such L_M the mappings L_S from (1.7), considered as operators only over $\mathbf{z} \in S^{(1)}, \mathbf{Z} \in S^{(2)}$ etc. $(p, q \geqslant 1)$, are not generally independent of the reference system. However, the tensors Δ, Λ from (1.4) occur in their definition (1.7), i.e., the L_S are parametrized by the tensors Δ, Λ, and then the L_S considered as mappings of the pair of tensors $\gamma = \{\mathbf{z}; \Delta\}$ or the triplet of tensors $\gamma = \{\mathbf{Z}; \Delta, \Lambda\}$ in $S^{(q)}$ etc., are independent of the reference system in the sense of (2.1).

This is seen from the following more general assertion (we henceforth limit ourselves to tensors of just the second rank $\mathbf{U}, \mathbf{Z}$).

Lemma 3. For the tensor processes $\mathbf{U} \in M^{(2)}$ and $\mathbf{Z} \in S^{(2)}$ let the relations (1.3)-(1.5) be satisfied with non-degenerate Δ, Λ and let parametrized tensors Δ, Λ of the mapping $L_0[\mathbf{U}; \Delta, \Lambda]$ and $L_1[\mathbf{Z}; \Delta, \Lambda]$ be given with values in a set of tensor processes of the second rank in any reference system related one-to-one by a relation of the type (1.7) and its inverse

$$L_1[\mathbf{Z}; \Delta, \Lambda] = \Delta L_0[\Delta^{-1}\mathbf{Z}\Lambda^{-1}; \Delta, \Lambda]\Lambda \qquad (2.3)$$
$$L_0[\mathbf{U}; \Delta, \Lambda] = \Delta^{-1}L_1[\Delta\mathbf{U}\Lambda; \Delta, \Lambda]\Lambda^{-1}$$

Then: 1) L_0 takes a value in $M^{(2)}$ if and only if L_1 takes a value in $S^{(2)}$); 2) L_0 and L_1 are independent of the reference system only simultaneously; 3) L_0 and L_1 are isotropic in the set of arguments only simultaneously; 4) if L_0 and L_1 are independent of the reference system and take values in $M^{(2)}$ and $S^{(2)}$, respectively, then they allow of the following representations (general reduced forms)

$$L_0[\mathbf{U}; \Delta, \Lambda] \equiv L_{00}[\mathbf{U}; \Xi, \Upsilon] \equiv \Xi^{-1}L_{10}[\Xi\mathbf{U}\Upsilon; \Xi, \Upsilon]\Upsilon^{-1} \qquad (2.4)$$
$$L_1[\mathbf{Z}; \Delta, \Lambda] \equiv \mathbf{Q}L_{10}[\mathbf{Q}^T\mathbf{Z}\mathbf{Q}; \Xi, \Upsilon]\mathbf{Q}^T \equiv \Delta L_{00}[\Delta^{-1}\mathbf{Z}\Lambda^{-1}; \Xi, \Upsilon]\Lambda$$

where L_{00} and L_{10} are invariant under shifts of the time argument.

Corollary 1. The mapping L_S from (1.7), parametrized by the tensors Δ, Λ is independent of the reference system only simultaneously with L_M.

Corollary 2. If $\Xi \equiv \Upsilon \equiv \mathbf{I}$, are selected in (1.5) under the conditions of Lemma 3, then the reduced forms (2.4) have the simple form

$$L_0[\mathbf{U}; \Delta, \Lambda] \equiv L_{00}[\mathbf{U}], \qquad (2.5)$$
$$L_1[\mathbf{Z}; \Delta, \Lambda] \equiv \mathbf{Q}L_{00}[\mathbf{Q}^T\mathbf{Z}\mathbf{Q}]\mathbf{Q}^T \equiv \mathbf{Q}L_{00}[\mathbf{U}]\mathbf{Q}^T$$
$$(L_{00}[\mathbf{U}] \equiv L_0[\mathbf{U}; \mathbf{I}, \mathbf{I}])$$

Example 1. Let

$$L_0[\mathbf{U}; \Delta, \Lambda] \equiv f_0(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)}; \Xi, \Upsilon)$$

be a tensor-valued function (of second rank) of the instantaneous values of the arguments. Then, by (2.3) and (2.4),

$$L_1[\mathbf{Z}; \Delta, \Lambda] \equiv \Delta f_0(\Delta^{-1}\mathbf{Z}\Lambda^{-1}, \Delta^{-1}D[\mathbf{Z}]\Lambda^{-1}, \ldots, \Delta^{-1}D[\mathbf{Z}]\Lambda^{-1}; \Xi, \Upsilon)\Lambda =$$
$$=: f_1(\mathbf{Z}, D[\mathbf{Z}], \ldots, D^k[\mathbf{Z}]; \Delta, \Lambda)$$

is also a tensor-valued function. In particular, for $\Xi \equiv \Upsilon \equiv \mathbf{I}$ we have (2.5) with

$$L_{00}[\mathbf{U}] \equiv f_0(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)}; \mathbf{I}, \mathbf{I}) =: f(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)})$$

where if f_0 and f_1 (meaning also f) are isotropic in the set of variables, then (2.5) takes the form

$$L_0[\mathbf{U}; \Delta, \Lambda] \equiv f(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)})$$
$$L_1[\mathbf{Z}; \Delta, \Lambda] \equiv f(\mathbf{Z}, \mathbf{Z}^{\circ}, \ldots, \mathbf{Z}^{\circ(k)})$$

where $\mathbf{Z}^{\circ}, \mathbf{Z}^{\circ\circ}, \ldots, \mathbf{Z}^{\circ(k)}$ are neutral derivatives /6, 7, 11-13, 15-17, 20/ up to the order of k inclusive.

Lemma 3 establishes a one-to-one correspondence between the sets of mappings L_0 and L_1 parametrized by the tensors Δ, Λ and their properties and yields their reduced forms.

Corollaries 1 and 2 show that the absence of parametrization of the mapping L_0 (when $L_0 = L_M$) does not remove parametrization of the mapping L_1 ($L_1 = L_S$) and does not result in any appreciable simplification of the properties (Lemma 2) or the general reduced forms (2.4) of the mappings $L_0 = L_M$ and $L_1 = L_S$. On the other hand, the absence of parametrization of the mapping L_1 independent of the reference system results in a much narrower subclass of mappings L_1 and their material analogues L_0, and reduced forms of a more special kind than (2.4).

Analogous concepts and assertions can be extended to the case when the correspondence between the tensor processes γ and π is given in implicit form, namely, in the form of an equation, generally, parametrized by a set χ of certain other tensor processes

$$H[\gamma(\tau),\ \pi(\tau);\ \chi(\tau)] = 0 \tag{2.6}$$

We say that (2.6) is independent of the reference system if it has one and the same set of solutions - sets of tensor processes $\{\gamma,\ \pi;\ \chi\}$, in all the reference systems, i.e., the kernel H is conserved

$$H = 0 \Leftrightarrow H_* = 0\ (\operatorname{Ker} H = \operatorname{Ker} H_*) \tag{2.7}$$

In the case of the unique solvability of (2.6) for π in the form of a certain mapping $L[\gamma;\ \chi]$ the definition (2.7) obviously corresponds to L being independent of the reference system in the sense of (2.1). On the other hand, if H is a mapping independent of the reference system in the sense of (2.1), then (2.6) is independent of the reference system in the sense of (2.7). Assertions analogous to those presented above for operators of the form (2.1) hold for Eqs.(2.6) and (2.7).

Lemma 4. For the tensor processes **U**, **P** and **Z**, **T** equivalent in the sense of (1.3), let $\mathbf{Z} = \boldsymbol{\Delta}_1\mathbf{U}\boldsymbol{\Lambda}_1$, $\mathbf{T} = \boldsymbol{\Delta}_2^{-1T}\mathbf{P}\boldsymbol{\Lambda}_2^{-1T}$ pairwise, where $\boldsymbol{\Delta}_l$, $\boldsymbol{\Lambda}_l$ $(l = 1, 2)$ are non-degenerate and satisfy the relations (1.4) and (1.5), and suppose we are given the equations $H_0[\mathbf{U}, \mathbf{P}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0$ and $H_1[\mathbf{Z}, \mathbf{T};\ \boldsymbol{\Delta}_l,\ \boldsymbol{\Lambda}_l] = 0$, connected by equivalence relations for the mentioned **U**, **P**, **Z**, **T**, i.e.

$$\begin{gathered}
H_1[\mathbf{Z}, \mathbf{T}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_0[\boldsymbol{\Delta}_1^{-1}\mathbf{Z}\boldsymbol{\Lambda}_1^{-1}, \boldsymbol{\Delta}_2{}^T\mathbf{T}\boldsymbol{\Lambda}_2{}^T; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \\
H_0[\mathbf{U}, \mathbf{P}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_1[\boldsymbol{\Delta}_1\mathbf{U}\boldsymbol{\Lambda}_1,\ \boldsymbol{\Delta}_2^{-1T}\mathbf{P}\boldsymbol{\Lambda}_2^{-1T}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0
\end{gathered} \tag{2.8}$$

Then 1) $\mathbf{U}, \mathbf{P} \in M^{(2)} \Leftrightarrow \mathbf{Z}, \mathbf{T} \in S^{(2)}$; 2) the equations $H_0 = 0$ and $H_1 = 0$ are independent of the reference system only simultaneously; 3) the equations $H_0 = 0$ and $H_1 = 0$ are isotropic in the set of their unknowns (and parameters) only simultaneously; 4) if $\mathbf{U}, \mathbf{P} \in M^{(2)}$, $\mathbf{Z}, \mathbf{T} \in S^{(2)}$ and the equations $H_0 = 0$ and $H_1 = 0$ are independent of the reference system, then they allow of the following reduced forms:

$$\begin{gathered}
H_0[\mathbf{U}, \mathbf{P}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_{00}[\mathbf{U}, \mathbf{P}; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l] = 0 \Leftrightarrow \\
H_{10}[\boldsymbol{\Xi}_1\mathbf{U}\boldsymbol{\Upsilon}_1, \boldsymbol{\Xi}_2^{-1T}\mathbf{P}\boldsymbol{\Upsilon}_2^{-1T}; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l] = 0 \\
H_1[\mathbf{Z}, \mathbf{T}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_{10}[\mathbf{Q}^T\mathbf{Z}\mathbf{Q}, \mathbf{Q}^T\mathbf{T}\mathbf{Q}; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l] = 0 \Leftrightarrow \\
H_{00}[\boldsymbol{\Delta}_1^{-1}\mathbf{Z}\boldsymbol{\Lambda}^{-1}, \boldsymbol{\Delta}_2{}^T\boldsymbol{\Lambda}_2{}^T; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l] = 0
\end{gathered} \tag{2.9}$$

where $H_{00} = 0$ and $H_{10} = 0$ are invariant under a shift of the time argument.

The assertions of Corollaries 1 and 2 are carried over analogously to the cases of (2.8) and (2.9). Thus for $\boldsymbol{\Xi}_l \equiv \boldsymbol{\Upsilon}_l \equiv \mathbf{I}$ we obtain (2.9) in the form

$$\begin{gathered}
H_0[\mathbf{U}, \mathbf{P}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_{00}[\mathbf{U}, \mathbf{P}] = 0 \\
H_1[\mathbf{Z}, \mathbf{T}; \boldsymbol{\Delta}_l, \boldsymbol{\Lambda}_l] = 0 \Leftrightarrow H_{00}[\mathbf{Q}^T\mathbf{Z}\mathbf{Q}, \mathbf{Q}^T\mathbf{T}\mathbf{Q}] = 0
\end{gathered} \tag{2.10}$$

Example 2. Just as in Example 1, we give the equation $H_0 = 0$ in the form of an ordinary differential equation

$$h_0(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)}, \mathbf{P}, \mathbf{P}^{\cdot}, \ldots, \mathbf{P}^{(m)}; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l) = 0$$

Then according to (2.8) and (2.9), we have the equation $H_1 = 0$ in the form

$$\begin{gathered}
h_0(\boldsymbol{\Delta}_1^{-1}\mathbf{Z}\boldsymbol{\Lambda}_1^{-1}, \boldsymbol{\Delta}_1^{-1}D_1[\mathbf{Z}]\boldsymbol{\Lambda}_1^{-1}, \ldots, \boldsymbol{\Delta}_1^{-1}D_1{}^k[\mathbf{Z}]\boldsymbol{\Lambda}_1^{-1} \\
\boldsymbol{\Delta}_2{}^T\mathbf{T}\boldsymbol{\Lambda}_2{}^T\boldsymbol{\Delta}_2{}^T D_2[\mathbf{T}]\boldsymbol{\Lambda}_2{}^T, \ldots, \boldsymbol{\Delta}_2{}^T D_2[\mathbf{T}]\boldsymbol{\Lambda}_2{}^T; \boldsymbol{\Xi}_l, \boldsymbol{\Upsilon}_l) = 0
\end{gathered}$$

i.e., after renotation

$$h_1(\mathbf{Z},\ D_1[\mathbf{Z}],\ \ldots,\ D_1{}^k[\mathbf{Z}],\ \mathbf{T},\ D_2[\mathbf{T}],\ \ldots,\ D_2{}^m[\mathbf{T}];\ \boldsymbol{\Delta}_l,\ \boldsymbol{\Lambda}_l) = 0$$

For $\boldsymbol{\Xi}_l \equiv \boldsymbol{\Upsilon}_l \equiv \mathbf{I}$ we have (2.10) with

$$H_{00}[\mathbf{U}, \mathbf{P}] \equiv h_0(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)}, \mathbf{P}, \mathbf{P}^{\cdot}, \ldots, \mathbf{P}^{(m)}; \mathbf{I}, \mathbf{I}) =: h(\mathbf{U}, \mathbf{U}^{\cdot}, \ldots, \mathbf{U}^{(k)}, \mathbf{P}, \mathbf{P}^{\cdot}, \ldots, \mathbf{P}^{(m)})$$

where, if the equations $h_0 = 0$ (and $h_1 = 0$), meaning also $h = 0$, are isotropic in the set of unknowns, then the equation $H_1 = 0$ takes the form
$h(\mathbf{Z}, \mathbf{Z}^{\circ}, \ldots, \mathbf{Z}^{\circ(k)}, \mathbf{T}, \mathbf{T}^{\circ}, \ldots, \mathbf{T}^{\circ(m)}) = 0$

3. Constitutive relationships (CR): explicit and implicit reduced forms. The fundamental principles; determinism and causality, locality, and independence of the reference system, are assumed in the mechanics of a continuous medium for constructing the CR connecting the characteristics of the stress and strain state (we confine ourselves to isothermal processes). The classical tensor measures of stress and strain of the second rank, reproducibile in principle in M-tests /1/ with homogeneous specimens and their states (we call the measures of stress σ and strain ε reproducibe in such tests macroscopic measures) are used as the stress and strain state characteristics in the classical theory of the CR stipulating satisfaction of the macrophysical determinacy hypotheses /1, 2/.

The strain affinor $\mathbf{A}$, the Cauchy stress tensor $\mathbf{S}$, different right and left measures including (1.9), and others can be used as the macroscopic measures of σ and ε. Any right tensor processes of the second rank $\sigma_0, \varepsilon_0 \in M^{(2)}$, whose histories up to any time t are uniquely and independently of the reference system related to the histories of the tensors Σ_{I} and $\mathbf{X}$ up to the time t, where ε_0 and $\mathbf{X}$ are connected independently

$$\begin{gathered}\varepsilon_0(\mathbf{a}, t) = \operatorname{def}[\mathbf{X}(\mathbf{a}, \tau)]_{\tau\leqslant t}, \quad \sigma_0(\mathbf{a}, t) = \operatorname{str}[\mathbf{X}(\mathbf{a}, \tau), \Sigma_{\mathrm{I}}(\mathbf{a}, \tau)]_{\tau\leqslant t} \\ \mathbf{X}(\mathbf{a}, t) = \operatorname{Def}[\varepsilon_0(\mathbf{a}, \tau)]_{\tau\leqslant t}, \quad \Sigma_{\mathrm{I}}(\mathbf{a}, t)\operatorname{Str}[\varepsilon_0(\mathbf{a}, \tau), \sigma_0(\mathbf{a}, t)]_{\tau\leqslant t}\end{gathered} \tag{3.1}$$

will be called right measures of stress and strain.

Any spatial analogues of the tensors σ_0, ε_0 constructed in the same way as (1.9) according to (1.3), (1.6), and (2.2)

$$\sigma_1 = \Delta_2^{-1T}\sigma_0\Lambda_2^{-1T}, \quad \varepsilon_1 = \Delta_1\varepsilon_0\Lambda_1 \tag{3.2}$$

will be called left measures of the stress and strain σ_1, ε_1.

Taking account of Lemmas 1-3 and the Corollaries (Sect.2), we obtain the following theorem.

Theorem 1. The CR of any medium has infinitely many general material representations (in terms of σ_0, ε_0) and spatial representations (in terms of σ_1, ε_1) in the form

$$\begin{gathered}\sigma_0(\mathbf{a}, t) = L_0[\varepsilon_0(\mathbf{a}, \tau)]_{\tau\leqslant t} \\ \sigma_1(\mathbf{a}, t) = L_1[\varepsilon_1(\mathbf{a}, \tau); \Delta_l(\mathbf{a}, \tau), \Lambda_l(\mathbf{a}, \tau)]_{\tau\leqslant t}\end{gathered} \tag{3.3}$$

where the relationships (1.3)-(1.6), (2.2), (3.1), (3.2) with non-degenerate Δ_l, Λ_l $(l = 1, 2)$, are satisfied, the mappings L_0, L_1 are independent of the reference system, take values, respectively, in the sets $\{\sigma_0\} \subset M^{(2)}$ and $\{\sigma_1\} \subset S^{(2)}$, are connected by the relationships

$$\begin{gathered}L_1[\varepsilon_1; \Delta_l, \Lambda_l]_{\tau\leqslant t} = \Delta_2^{-1T}L_0[\Delta_1^{-1}\varepsilon_1\Lambda_1^{-1}]_{\tau\leqslant t}\Lambda_2^{-1T} \\ L_0[\varepsilon_0]_{\tau\leqslant t} = \Delta_2{}^T L_1[\Delta_1\varepsilon_0\Lambda_1; \Delta_l, \Lambda_l]_{\tau\leqslant t}\Lambda_2{}^T\end{gathered} \tag{3.4}$$

and allow of the following general reduced forms:

$$\begin{gathered}L_0[\varepsilon_0]_{\tau\leqslant t} = F_0[\varepsilon_0]_{\tau\leqslant t} \equiv \Xi_2{}^T F_1[\Xi_1\varepsilon_0\Upsilon_1]_{\tau\leqslant t}\Upsilon_2{}^T \\ L_1[\varepsilon_1; \Delta_l, \Lambda_l]_{\tau\leqslant t} = \mathbf{Q}F_1[\mathbf{Q}^T\varepsilon_1\mathbf{Q}]_{\tau\leqslant t}\mathbf{Q}^T \equiv \Delta_2^{-1T}F_0[\Delta_1^{-1}\varepsilon_1\Lambda_1^{-1}]_{\tau\leqslant t}\Lambda_2^{-1T}\end{gathered} \tag{3.5}$$

with mappings F_0 and F_1 invariant with respect to the shift of the time argument.

Corollary 3. The CR of the Il'yushin macroscopic determinacy postulate and the Noll CR

$$\Sigma_{\mathrm{I}} = \Phi[\Psi_{\mathrm{I}}]_{\tau\leqslant t}, \quad \mathbf{S} = \mathbf{Q}R[\mathbf{X}]_{\tau\leqslant t}\mathbf{Q}^T \tag{3.6}$$

are mutually equivalent, are general forms of the form (3.5) and describe the properties of any medium.

Analogously, taking account of Lemma 4 the following theorem holds.

Theorem 2. Implicit assignment of the CR of any medium has infinitely many material and spatial representations in the form of the equations

$$\begin{gathered}H_0[\sigma_0(\mathbf{a}, \tau), \varepsilon_0(\mathbf{a}, \tau)]_{\tau\leqslant t} = 0 \\ H_1[\sigma_1(\mathbf{a}, \tau), \varepsilon_1(\mathbf{a}, \tau); \Delta_l(\mathbf{a}, \tau), \Lambda_l(\mathbf{a}, \tau)]_{\tau\leqslant t} = 0 \ (l = 1, 2)\end{gathered} \tag{3.7}$$

independent of the reference system, connected by the equivalence relationships

$$H_1[\sigma_1, \varepsilon_1; \Delta_l, \Lambda_l]_{\tau\leqslant t} = 0 \Leftrightarrow H_0[\Delta_2{}^T\sigma_1\Lambda_2{}^T, \Delta_1^{-1}\varepsilon_1\Lambda_1^{-1}]_{\tau\leqslant t} = 0 \tag{3.8}$$
$$H_0[\sigma_0, \varepsilon_0]_{\tau\leqslant t} = 0 \Leftrightarrow H_1[\Delta_2^{-1T}\sigma_0\Lambda_2^{-1T}, \Delta_1\varepsilon_0\Lambda_1; \Delta_l, \Delta_l]_{\tau\leqslant t} = 0$$

(when (1.3)-(1.6), (2.2), (3.1), (3.2) are taken into account) and allowing the following general reduced forms

$$H_0[\sigma_0, \varepsilon_0]_{\tau\leqslant t} = 0 \Leftrightarrow N_0[\sigma_0, \varepsilon_0]_{\tau\leqslant t} = 0 \Leftrightarrow N_1[\Xi_2^{-1T}\sigma_0\Upsilon_2^{-1T}, \ \Xi_1\varepsilon_0\Upsilon_1]_{\tau\leqslant t} = 0 \tag{3.9}$$
$$H_1[\sigma_1, \varepsilon_1; \Delta_l, \Lambda_l]_{\tau\leqslant t} = 0 \Leftrightarrow N_1[\mathbf{Q}^T\sigma_1\mathbf{Q}, \mathbf{Q}^T\varepsilon_1\mathbf{Q}]_{\tau\leqslant t} = 0 \Leftrightarrow$$
$$\Leftrightarrow N_0(\Delta_2{}^T\sigma_1\Lambda_2{}^T, \Delta_1^{-1}\varepsilon_1\Lambda_1^{-1}]_{\tau\leqslant t} = 0$$

with equations $N_0 = 0$ and $N_1 = 0$ that are invariant with respect to a shift in the time argument.

Like (3.6) the implicit Il'yushin and Noll general CR forms have the form

$$N_{0I}[\Sigma_I, \Psi_I]_{\tau\leqslant t} = 0, \quad N_{1N}[\mathbf{Q}^T\mathbf{S}\mathbf{Q}, \mathbf{X}]_{\tau\leqslant t} = 0 \tag{3.10}$$

Corollary 4. The implicit material and spatial CR representations (3.7)-(3.9) are solvable in the form of explicit material and spatial representations for $\sigma_0(\mathbf{a}, t)$ and $\sigma_1(\mathbf{a}, t)$ (the principles of determinism and causality are satisfied) only simultaneously and, moreover, in the form (3.3)-(3.5), respectively.

Theorems 1 and 2 and Corollary 4 reveal the possibility of realizing the fundamental principles of the classical theory of CR of the mechanics of a continuous medium and yield answers to the fundamental questions posed at the beginning of the paper.

4. *Examples.* Theorems 1 and 2 are also illustrated by Examples 1 and 2 in Sect.2 and enable one to change from material CR representations to spatial representations and vice-versa, moreover, with different Δ_l, Λ_l $(l = 1, 2)$.

Thus be setting $\sigma_0 = \Sigma_I, \varepsilon_0 = \mathbf{C}, \sigma_1 = \mathbf{S}, \varepsilon_1 = \mathbf{F}$, where $\Delta_1 = \mathbf{Q}, \Lambda_1 = \mathbf{Q}^T, \Delta_2 = \mathbf{A}^{-1T}, \Lambda_2 = A^{-1}$, according to Theorem 1 we obtain the CR of a non-linearly elastic isotropic body /10/ in the material representation (in terms of $\Sigma_I, \mathbf{C}$) and the spatial representation (in terms of $\mathbf{S}, \mathbf{F}$)

$$\Sigma_I = f_0\mathbf{C}^{-1} + f_1\mathbf{I} + f_2\mathbf{C}, \quad \mathbf{S} = f_0\mathbf{I} + f_1\mathbf{F} + f_2\mathbf{F}^2 \tag{4.1}$$

and correspondingly in Lagrange and Euler components /1/ (the initial Lagrange and Euler coordinate systems are rectangular Cartesian)

$$S^{ij} = f_0g^{ij} + f_1\delta_{ij} + f_2g_{ij}, \quad \sigma_{ij} = f_0\delta_{ij} + f_1g_0{}^{ij} + f_2g_0^{ik}g_0^{kj} \tag{4.2}$$
$$\left(g_{ij} = \frac{\partial x_m}{\partial a_i}\frac{\partial x_m}{\partial a_j}, \quad g^{ij} = \frac{\partial a_i}{\partial x_m}\frac{\partial a_j}{\partial x_m}, \quad g_0{}^{ij} = \frac{\partial x_i}{\partial a_m}\frac{\partial x_j}{\partial a_m}\right)$$

where f_0, f_1, f_2 are functions of invariants of the tensor $\mathbf{C}$ (or $\mathbf{F}$), and x_i, a_i are Cartesian components of the vectors x, a (from (1.1)). Setting $\sigma_0 = \Sigma_I, \varepsilon_0 = \Psi_I \equiv {}^1/_2(\mathbf{C} - \mathbf{I}), \sigma_1 = \mathbf{S}, \varepsilon_1 = \mathbf{E}_1 \equiv {}^1/_2(\mathbf{I} - \mathbf{F}^{-1})$ and $\Delta_1 = \mathbf{A}^{-1T}, \Lambda_1 = \mathbf{A}^{-1}, \Delta_2 = \mathbf{A}^{-1T}, \Lambda_2 = \mathbf{A}^{-1}$ respectively, and taking account of (1.9), we obtain both CR representations of the theory of plasticity of small curvature (initially given in /30, 31/ in a spatial representation)

$$\Sigma_I - \sigma\mathbf{C}^{-1} = k(\mathbf{C}^{-1}\Psi_I{}^{\cdot} - v\mathbf{I})\mathbf{C}^{-1}, \quad \mathbf{S} - \sigma\mathbf{I} = k(\mathbf{V} - v\mathbf{I}) \tag{4.3}$$
$$(k = 2\Phi(s)/3v_u)$$

or in the Lagrange and Euler components, respectively

$$S^{ij} - \sigma g^{ij} = -k({}^1/_2 g^{\cdot ij} + vg^{ij}), \quad \sigma_{ij} - \sigma\delta_{ij} = k(v_{ij} - v\delta_{ij}) \tag{4.4}$$

Here Φ is a known material function, v_{ij} are the Euler components of the strain rate tensor $\mathbf{V}$, while the mean strain rate v, the strain rate intensity v_u, and the strain trajectory arc length s are expressed in terms of invariants of both the tensor $\mathbf{V}$ and the tensor $\mathbf{C}$ and its derivative, and the hydrostatic stress σ is expressed in terms of both $\mathbf{S}$ and Σ_I together with $\mathbf{C}$.

A formulation of a viscous fluid CR in terms of the same tensors and their components is analogous to (4.3) and (4.4).

Theorem 2 also enables a four term CR with a Jaumann derivative given in terms of spatially oriented tensors $\mathbf{S}$, $\mathbf{V}$ to be written equivalently in terms of materially oriented tensors Σ_I, C

$$\Sigma_I^{\cdot} + {}^1/_2 C^{-1}C^{\cdot}\Sigma_I + {}^1/_2\Sigma_I C^{\cdot}C^{-1} = a\Sigma_I - {}^1/_2 b\,(C^{-1})^{\cdot} + cC^{-1} \tag{4.5}$$
$$S^* = aS + bV + cI$$

and in components

$$S^{\cdot ij} + {}^1/_2 g^{ik}\dot{g}_{km}S^{mj} + {}^1/_2 g^{jm}\dot{g}_{mk}S^{ik} = aS^{ij} - {}^1/_2 b g^{\cdot ij} + cg^{ij} \tag{4.6}$$
$$\frac{d\sigma_{ij}}{dt} - \omega_{ik}\sigma_{kj} + \sigma_{ik}\omega_{kj} = a\sigma_{ij} + bv_{ij} + c\delta_{ij}$$

where $(\cdot)^*$ is the Jaumann derivative, ω_{ij} are the components of the angular velocity tensor (spin), and a, b, c are functions (functionals) or arbitrary (separate and combined) invariants of the tensors S and V (or Σ_I and C with its derivative). Representations of the CR of the theory of plastic flow (small curvature) (4.3), (4.4), the CR of linear hydroelasticity /3, 25/, three-term CR of plasticity /1, 32/, taken for finite strains in terms of the space tensors S and V are obtained as special cases of (4.5) and (4.6).

It is important to note that, as Corollary 4, Example 2 in Sect.2 and the present particular example of (4.5) and (4.6) show, the question of the unique solvability of equations of the form (3.7) with spatial measures $\sigma \equiv \sigma_1, \varepsilon \equiv \varepsilon_1$ for $\sigma_1(\mathbf{a}, t)$, i.e., when the determinism and causality principle are satisfied, reduces to the question of the solvability of the corresponding equation for the right measures σ_0, ε_0. In the case of Example 2 of Sect.2 and the CR (4.5) and (4.6), the solution of the system of differential (differential-functional) equations with parametrized differential operators (the Jaumann derivative, say) (parametrized tensors (1.4) determined according to (1.6) and (2.2) by unknown (arbitrary) spatial motion of an element of the medium) reduces to the solution of a system of ordinary differential (differential-functional) equations with non-parametrized operators of material differentiation with respect to time of the material tensors, which is a much simpler problem. The solution obtained for the material (right) tensor measures enables us to write the solution of the initial Eq.(3.7) rapidly for the spatial (left) measures according to (3.5).

Thus, for the model of a viscoelastic body of Maxwell type under finite strains given in terms of the left tensors S and V by an equation of the type (3.7)

$$D\,[S] = EV - T^{-1}S \tag{4.7}$$

(the operator $D\,[S]$ is determined according to (1.8), say, with $\Delta \equiv \Lambda^T \equiv \Lambda^{-1T}$, here, and E, T are constants) such a procedure yields an explicit form of the solution

$$S = EE_I - \frac{E}{T}\Lambda^{-1T}\left(\int_0^t \Lambda^T E_I \Lambda \exp\left(-\frac{t-\tau}{T}\right)d\tau\right)\Lambda^{-1} \tag{4.8}$$
$$E_I = {}^1/_2\,(I - F^{-1})$$

where E_I is the Almansi strain tensor.

5. *Conclusions*. The concepts of right and left objective tensors and the equivalence relations (1.3) between them etablish a one-to-one connection between mappings of the right and left tensors utilized, respectively, in the Lagrange and Euler descriptions of the characteristics of the motion and mechanical properties of the medium. The concepts of the mappings and equations being independent of the reference system afford effective opportunities for constructing general reduced forms of such mappings and equations connecting the objective tensors of different kinds, including reduced forms of the constitutive relationships (equations of state) and their "Lagrange" and "Euler" formulations.

By using these concepts for the classical mechanics of a continuous medium that stipulates satisfaction of the fundamental principles and hypotheses of macrophysical determinacy with respect to the CR, mentioned in Sect.3, an exhaustive set of general reduced CR forms (each of which corresponds to a specific choice of the stress and strain measures) is obtained, in both explicit and implicit form, and it is shown that the Il'yushin and Noll classical CR postulates macroscopic determinacy are equivalent general reduced forms. In turn, the specific selection of the stress and strain measures can dictate either the convenience of the computations or the processing and interpretation of the experimental data, or a tendency to satisfy definite regularities or dependences (of the proposed kind) for the class of materials (processes) under consideration, i.e., such a choice can be an essential element in constructing a theory of CR.

The question of the single-valued solvability of the equations (implicit forms) given in terms of the left tensors and, as a rule, parametrized by tensors governing the unknown (arbitrary) spatial motion of an element of the medium, reduces to the solvability of corresponding non-parametrized equations for the right strain and stress measures (in particular, parametrized differential equations with objective left tensor derivatives reduced to ordinary differential equations in the right tensors) by using the developed apparatus of equivalent representations.

The results obtained can be extended to tensors of higher rank ($r > 2$). Investigation of the special structure (reduced form) of non-parametrized mappings of left tensors into left, and the structures of mappings of tensors of other kinds including the mapping of tensors of different ranks, is of interest. Additional forms for describing mappings (specified explicitly and implicitly) can be given by an investigation of tensors of "mixed" type ((1.4) for a tensor of second rank) and their connections with the right and left tensors considered.

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SUPERSONIC CLEAVAGE OF AN ELASTIC STRIP*

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The problem of the longitudinal cleavage of an infinite elastic strip by a thin smooth rigid wedge is examined. The wedge moves symmetrically with respect to the faces of the strip at a constant supersonic velocity. Formulas are obtained that govern the stresses in the domain of wedge contact with the elastic medium and the displacements of points of the slit edge outside the contact domain for certain relationships between the parameters of the problem. Conditions are set up for which separation of the medium from the wedge surface occurs. Unlike the case of wedge motion at a speed less than the Rayleigh velocity /1, 2/, when a crack is formed ahead of the wedge, no crack is formed when the wedge moves at supersonic speed. The contact problem of the motion of a rigid stamp with a flat smooth base at a supersonic speed over the surface of an elastic strip was investigated /3/ in a similar formulation.

1. We will first consider the auxiliary problem (plane deformation) of the motion of a concentrated force P at a constant supersonic velocity V ($V > c_1 > c_2$, where c_1 and c_2 are, respectively, the velocity of sound of longitudinal and transverse waves in the elastic medium) over the surface of an elastic strip of thickness h. Let the strip be rigidly clamped along the base. Then the boundary conditions of the auxiliary problem in a moving system of coordinates whose origin is superposed on the point of application of the concentrated force, will have the form ($\delta(x)$ is the delta-function)

$$\sigma_y = -P\delta(x), \quad \tau_{xy} = 0 \ (y = 0), \ u = v = 0 \ (y = -h) \tag{1.1}$$

It is well-known that such a problem reduces to finding two wave functions connected by the boundary conditions and can be solved in closed form /3/. For the system of shock waves shown in Fig.1 we present the final expression for the displacement of points of the strip upper boundary in the direction of the y-axis

$$v(x, 0) = P\Theta^{-1}[\Pi(x) - D_1\Pi(x + 2\beta h) - D_2\Pi(x + \beta h + \gamma h)] \tag{1.2}$$

$$\Pi(t) = \begin{cases} 0 & (t > 0) \\ -1 & (t < 0) \end{cases}, \quad \Theta = G\frac{4\gamma(B+1)}{\gamma^2+1}$$

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